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S-matrix for a spin-$\frac{1}{2}$ particle in a Coulomb and scalar potential

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Abstract

The S-matrix for a spin-$\frac{1}{2}$ particle in the presence of a potential which is the sum of the Coulomb potential $V_c = -A_1/r$ and a Lorentz scalar potential $V_s = -A_2/r$ is calculated.

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1. Introduction

We consider a spin-$\frac{1}{2}$ particle in a potential which is the sum of the Coulomb potential $V_c = -A_1/r$ and a Lorentz scalar potential $V_s = -A_2/r$. The scalar potential is added to the mass term in the Dirac equation and may be interpreted as an effective position-dependent mass. If the scalar potential is assumed to be created by the exchange of massless scalar mesons, it has the form $V_s = -\frac{A_2}{r}$.

Exact solutions for the bound states in this mixed potential can be obtained by separation of variables [1, 2]. In this paper, we consider the scattering problem for such a potential which does not seem to have been treated in the literature. We calculate the phase shifts by the conventional technique and show how the scattering problem can also be solved algebraically.

This paper is organized as follows. In section 2 we separate variables in the Dirac equation obtaining the radial equations, in section 3 we solve the radial equations for the scattering problem and calculate the phase shifts and in section 4 we apply an algebraic technique to obtain the phase shifts. Section 5 contains the conclusions.

2. Separation of variables in the Dirac equation

The time-independent Dirac equation in the presence of the mixed potential may be written as

$$\left(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta\left(M - \frac{A_2}{r}\right) - \left(E + \frac{A_1}{r}\right)\right)\Psi(\mathbf{x}) = \mathbf{0} \tag{1}$$

where $\mathbf{p} = -\mathrm{i}\frac{\partial}{\partial \mathbf{x}}$ and $r = |\mathbf{x}|$. To separate variables we write $\Psi(\mathbf{x})$ in terms of two component spinors

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{2}$$

The two component angular solutions are eigenfunctions of J^2, J_z, L^2, S^2 and are of two types

$$\phi^{(+)}{}_{j,m} = \begin{pmatrix} \left(\frac{l+\frac{1}{2}+m}{2l+1}\right)^{\frac{1}{2}} Y_{l,m-\frac{1}{2}} \\ \left(\frac{l+\frac{1}{2}-m}{2l+1}\right)^{\frac{1}{2}} Y_{l,m+\frac{1}{2}} \end{pmatrix} \tag{3}$$

for $j = l + \frac{1}{2}$ and

$$\phi^{(-)}{}_{j,m} = \begin{pmatrix} \left(\frac{l+\frac{1}{2}-m}{2l+1}\right)^{\frac{1}{2}} Y_{l,m-\frac{1}{2}} \\ -\left(\frac{l+\frac{1}{2}+m}{2l+1}\right)^{\frac{1}{2}} Y_{l,m+\frac{1}{2}} \end{pmatrix} \tag{4}$$

for $j = l - \frac{1}{2}$.

In the above basis one can verify that

$$J^2\phi^{(\pm)}{}_{j,m} = j(j+1)\phi^{(\pm)}{}_{j,m} \tag{5}$$

$$(1 + \boldsymbol{\sigma}\cdot\mathbf{L})\phi^{(\pm)}{}_{j,m} = -\kappa\phi^{(\pm)}{}_{j,m} \tag{6}$$

where $\kappa = \pm\left(j + \frac{1}{2}\right)$ for $j = l \mp \frac{1}{2}$. Next, we put

$$\Psi = \begin{pmatrix} \frac{\mathrm{i}G_{lj}}{r}\phi^{l}{}_{jm} \\ \frac{F_{lj}}{r}\boldsymbol{\sigma}\cdot\mathbf{n}\phi^{l}{}_{jm} \end{pmatrix} \tag{7}$$

where $\mathbf{n} = \frac{\mathbf{r}}{r}$ and

$$\phi^{l}{}_{jm} = \phi^{(\pm)}{}_{j,m} \tag{8}$$

for $j = l \pm \frac{1}{2}$. Defining the Dirac operator

$$K = \gamma^0(1 + \boldsymbol{\Sigma}\cdot\mathbf{L}) \tag{9}$$

we have

$$K\Psi = -\kappa\Psi \tag{10}$$

where we have used the relation

$$[1 + \boldsymbol{\sigma}\cdot\mathbf{L}, \boldsymbol{\sigma}\cdot\mathbf{n}]_+ = 0. \tag{11}$$

Next, using the relations

$$\boldsymbol{\sigma}\cdot\mathbf{p}\frac{f(r)}{r}\phi^{l}{}_{jm} = -\frac{\mathrm{i}}{r}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\kappa f}{r}\right)\boldsymbol{\sigma}\cdot\mathbf{n}\phi^{l}{}_{jm} \tag{12}$$

and

$$\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{n}\frac{f(r)}{r}\phi^{l}{}_{jm} = -\frac{\mathrm{i}}{r}\left(\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{\kappa f}{r}\right)\phi^{l}{}_{jm} \tag{13}$$

we get the radial equations

$$\begin{aligned} \frac{\mathrm{d}G_{lj}}{\mathrm{d}r} + \frac{\kappa}{r}G_{lj} - \left(E + M - \frac{A_2}{r} + \frac{A_1}{r}\right)F_{lj} = 0 \\ \frac{\mathrm{d}F_{lj}}{\mathrm{d}r} - \frac{\kappa}{r}F_{lj} + \left(E - M + \frac{A_2}{r} + \frac{A_1}{r}\right)G_{lj} = 0. \end{aligned} \tag{14}$$

Next, let $\rho = kr$ where $k^2 = E^2 - M^2$. Then the radial equations take the form (we omit the indices l, j)

$$\frac{\mathrm{d}G}{\mathrm{d}\rho} = \left(\frac{E+M}{k} + \frac{A_1 - A_2}{\rho}\right) F - \frac{\kappa}{\rho} G \qquad \frac{\mathrm{d}F}{\mathrm{d}\rho} = -\left(\frac{E-M}{k} + \frac{A_1 + A_2}{\rho}\right) G + \frac{\kappa}{\rho} F. \quad (15)$$

3. Direct calculation of the phase shifts

In this section we solve the radial equation by using a technique similar to that used by Lin [3]. Defining the functions $u(\rho)$ and $v(\rho)$ by

$$G = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\rho}(u+v) \qquad F = \frac{\mathrm{i}}{2}\left(\frac{E-M}{E+M}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\rho}(u-v) \quad (16)$$

we get

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = \frac{\mathrm{i}E\gamma}{k\rho} u + \frac{1}{\rho}\left(-\kappa + \frac{\mathrm{i}M\gamma'}{k}\right) v \qquad \frac{\mathrm{d}v}{\mathrm{d}\rho} = -\left(\frac{\kappa}{\rho} + \mathrm{i}\frac{M\gamma'}{k\rho}\right) u - \mathrm{i}\frac{\gamma E}{k\rho} v - 2\mathrm{i}v \quad (17)$$

where $\gamma = A_1 + \frac{M}{E} A_2$ and $\gamma' = A_1 + \frac{E}{M} A_2$. Eliminating v we get

$$\rho \frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} + (1 + 2\mathrm{i}\rho)\frac{\mathrm{d}u}{\mathrm{d}\rho} + \left(\frac{2E\gamma}{k} - \frac{\lambda^2}{\rho}\right) u = 0 \quad (18)$$

where $\lambda = \frac{\kappa}{|\kappa|}\left(\kappa^2 - A_1{}^2 + A_2{}^2\right)^{\frac{1}{2}}$. Let

$$u(\rho) = \rho^{|\lambda|} w(\rho). \quad (19)$$

Then $w(\rho)$ satisfies the equation

$$\rho \frac{\mathrm{d}^2 w}{\mathrm{d}\rho^2} + (2|\lambda| + 1 + 2\mathrm{i}\rho)\frac{\mathrm{d}w}{\mathrm{d}\rho} + 2\mathrm{i}\left(|\lambda| - \frac{\mathrm{i}E\gamma}{k}\right) w = 0. \quad (20)$$

Restoring the indices l, j we get

$$u_{lj} = a_{lj}\rho^{|\lambda|}\Phi\left(|\lambda| - \frac{\mathrm{i}E\gamma}{k}, 2|\lambda| + 1, -2\mathrm{i}\rho\right) \quad (21)$$

where a_{lj} is a constant and $\Phi(a, b, z)$ is the confluent hypergeometric function. Using equation (17) it is easy to show that

$$v_{lj} = a_{lj}\frac{|\lambda| - \frac{\mathrm{i}E\gamma}{k}}{-\kappa + \frac{\mathrm{i}M\gamma'}{k}}\rho^{|\lambda|}\Phi\left(|\lambda| - \frac{\mathrm{i}E\gamma}{k} + 1, 2|\lambda| + 1, -2\mathrm{i}\rho\right). \quad (22)$$

For $r \to \infty$ we get

$$G^{\mathrm{out}} = \frac{a_{lj}}{2^{|\lambda|+1}}\,\mathrm{e}^{-\frac{E\gamma\pi}{2k}}\,\mathrm{e}^{\mathrm{i}kr + \mathrm{i}\frac{E\gamma}{k}\ln 2kr}\frac{\Gamma(2|\lambda|+1)}{\Gamma\left(1 + |\lambda| + \mathrm{i}\frac{E\gamma}{k}\right)}$$

$$G^{\mathrm{in}} = \frac{a_{lj}}{2^{|\lambda|+1}}\,\mathrm{e}^{-\frac{E\gamma\pi}{2k}}\,\mathrm{e}^{-\mathrm{i}kr - \mathrm{i}\frac{E\gamma}{k}\ln 2kr}\frac{\Gamma(2|\lambda|+1)}{\Gamma\left(1 + |\lambda| - \mathrm{i}\frac{E\gamma}{k}\right)}\,\mathrm{e}^{\mathrm{i}\pi|\lambda|}\frac{|\lambda| - \mathrm{i}\frac{E\gamma}{k}}{-\kappa + \mathrm{i}\frac{M\gamma'}{k}}. \quad (23)$$

Thus the phase shifts are given by

$$\mathrm{e}^{2\mathrm{i}\delta_{l,j}(k)} = \frac{-\kappa + \mathrm{i}\frac{M\gamma'}{k}}{|\lambda| - \mathrm{i}\frac{E\gamma}{k}}\frac{\Gamma\left(1 + |\lambda| - \mathrm{i}\frac{E\gamma}{k}\right)}{\Gamma\left(1 + |\lambda| + \mathrm{i}\frac{E\gamma}{k}\right)}\,\mathrm{e}^{-\mathrm{i}\pi\lambda}. \quad (24)$$

4. Algebraic calculation of the phase shifts

In this section we apply an algebraic technique to calculate the phase shifts. This technique was used earlier by Alhassid *et al* [4] for the relativistic Coulomb problem.

With $\Phi = \binom{G_{lj}}{F_{lj}}$ we get

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}(\kappa\rho_3 + A_2\rho_1 - \mathrm{i}A_1\rho_2) - M\rho_1 - \mathrm{i}E\rho_2\right]\Phi = 0 \tag{25}$$

where ρ_i are the Pauli matrices.

The potential matrix $\Lambda = A_2\rho_1 - \mathrm{i}A_1\rho_2 + \kappa\rho_3$ may be diagonalized by using the result

$$\mathrm{e}^{\mathrm{i}\beta\rho_2}\,\mathrm{e}^{-\alpha\rho_1}\,\Lambda\,\mathrm{e}^{\alpha\rho_1}\,\mathrm{e}^{-\mathrm{i}\beta\rho_2} = \lambda\rho_3 \tag{26}$$

where

$$\tanh 2\alpha = \frac{A_1}{\kappa} \qquad \tan 2\beta = \frac{A_2}{\kappa'} \qquad \kappa' = \epsilon(\kappa)\left(\kappa^2 - A_1{}^2\right)^{\frac{1}{2}} \qquad \lambda = \epsilon(\kappa)\left(\kappa'^2 + A_2{}^2\right)^{\frac{1}{2}} \tag{27}$$

where $\epsilon(\kappa) = \frac{\kappa}{|\kappa|}$. Defining

$$\Phi = \mathrm{e}^{\alpha\rho_1}\,\mathrm{e}^{-\mathrm{i}\beta\rho_2}\,\hat{\Phi} \tag{28}$$

we get

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{r}\rho_3 + \mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}\right]\hat{\Phi} = 0 \tag{29}$$

where

$$\mathrm{i}k_1 = \frac{EA_1A_2 - M\kappa'^2}{\kappa'\lambda} \qquad \mathrm{i}k_2 = -\mathrm{i}E\frac{\kappa}{\kappa'} \qquad \mathrm{i}k_3 = -\frac{(MA_2 + EA_1)}{\lambda}. \tag{30}$$

One can verify that

$$E^2 = \mathbf{k}^2 + M^2. \tag{31}$$

Multiplying on the left by the operator $\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\lambda}{r}\rho_3 - \mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}$ we get the second-order equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\lambda}{r^2}\rho_3 - \frac{\lambda^2}{r^2} + \frac{2E\gamma}{r} + k^2\right]\hat{\Phi} = 0 \tag{32}$$

where

$$\gamma = A_1 + \frac{M}{E}A_2. \tag{33}$$

For $r \to \infty$, omitting the $\frac{1}{r}$ terms in equation (25), we have two types of free particle solutions which satisfy the conditions

$$(M\rho_1 + \mathrm{i}E\rho_2)\Phi^{\mathrm{in}} = -\mathrm{i}k\Phi^{\mathrm{in}} \qquad (M\rho_1 + \mathrm{i}E\rho_2)\Phi^{\mathrm{out}} = \mathrm{i}k\Phi^{\mathrm{out}} \tag{34}$$

where $k^2 = E^2 - M^2$. The explicit form of these solutions is

$$\Phi^{\mathrm{in}} = g^{\mathrm{in}}\begin{pmatrix}1\\ \frac{\mathrm{i}k}{E+M}\end{pmatrix}\mathrm{e}^{-\mathrm{i}kr} \qquad \Phi^{\mathrm{out}} = g^{\mathrm{out}}\begin{pmatrix}1\\ \frac{-\mathrm{i}k}{E+M}\end{pmatrix}\mathrm{e}^{\mathrm{i}kr}. \tag{35}$$

To calculate the scattering matrix we need to find the ratio $\frac{g^{\mathrm{out}}}{g^{\mathrm{in}}}$.

Next, equations (29) and (30) give, in the limit $r \to \infty$,

$$\frac{g^{\mathrm{out}}}{g^{\mathrm{in}}} = \frac{\hat{g}^{\mathrm{out}}}{\hat{g}^{\mathrm{in}}}\,\frac{\mathrm{i}k_-S_{11} - \mathrm{i}k_3S_{12} - \mathrm{i}kS_{12}}{\mathrm{i}k_-S_{11} - \mathrm{i}k_3S_{12} + \mathrm{i}kS_{12}} \tag{36}$$

where $S = e^{\alpha\rho_1} e^{-i\beta\rho_2}$. After considerable effort we get

$$
\begin{aligned}
(ik_- S_{11} - ik_3 S_{12}) &= -\frac{E+M}{A_1 - A_2}(\kappa + \lambda)\frac{(A_1(\kappa' + \lambda) - A_2(\kappa' + \kappa))}{4\kappa'\lambda \cosh\alpha, \cos\beta} \\
-ikS_{12} &= -ik\frac{(A_1(\kappa' + \lambda) - A_2(\kappa' + \kappa))}{4\kappa'\lambda \cosh\alpha \cos\beta}.
\end{aligned}
\tag{37}
$$

Thus we get

$$
\frac{ik_- S_{11} - ik_3 S_{12} - ikS_{12}}{ik_- S_{11} - ik_3 S_{12} + ikS_{12}} = \frac{\kappa + \lambda + \frac{i}{k}(E\gamma - M\gamma')}{\kappa + \lambda - \frac{i}{k}(E\gamma - M\gamma')}
\tag{38}
$$

where we have used

$$
E\gamma - M\gamma' = (E - M)(A_1 - A_2).
\tag{39}
$$

Finally, using the relation

$$
\frac{\kappa - i\frac{M\gamma'}{k}}{\lambda - i\frac{E\gamma}{k}} = \frac{\lambda + i\frac{E\gamma}{k}}{\kappa + i\frac{M\gamma'}{k}}
\tag{40}
$$

we get

$$
\frac{g^{\text{out}}}{g^{\text{in}}} = \frac{\hat{g}^{\text{out}}}{\hat{g}^{\text{in}}}\frac{\kappa - i\frac{M\gamma'}{k}}{\lambda - i\frac{E\gamma}{k}}.
\tag{41}
$$

One may calculate the phase shifts by an algebraic technique by observing that the differential equation for $\hat{g}$ obtained from equation (32) is of the same form as the radial equation for the two-dimensional hydrogen atom with the Hamiltonian

$$
H = \frac{\mathbf{p}^2}{2} - \frac{\alpha}{r}.
\tag{42}
$$

The latter problem has been treated by Alhassid *et al* [5]. They use the fact that the problem has dynamical symmetry under the $SO(2,1)$ group so that the asymptotic states obey recursion relations under the action of the generators of the $SO(2,1)$ algebra. On the other hand, the asymptotic states are linear superpositions of eigenstates of the Casimir operator of the algebra of the Euclidean group $E(2)$. As a consequence one gets a recursion relation for the ratio of the out and in amplitudes which can be solved. Making appropriate changes in their results: $\alpha \to 2E\gamma$ and $m \to \lambda + \frac{1}{2}$, we get

$$
\frac{\hat{g}^{\text{out}}}{\hat{g}^{\text{in}}} = -e^{i\pi\lambda}\frac{\Gamma\left(1 + \lambda - i\frac{E\gamma}{k}\right)}{\Gamma\left(1 + \lambda + i\frac{E\gamma}{k}\right)}.
\tag{43}
$$

Using equation (43) in equation (41) we have the same phase shifts as those of equation (24).

5. Conclusion

In conclusion we have shown that the phase shifts for the generalized Dirac equation can be found algebraically. The results are in agreement with the conventional calculation. The advantage of the algebraic method is that we work in the $r \to \infty$ limit instead of solving the radial functions and then taking the limit.

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